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A Solution Spectrum of the Nonlinear Schrödinger Equation. II

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It was shown in a previous communication that the nonlinear Schrödinger equation exhibits a spectrum of eigenfunctions of the form $\Psi = \sum_{k'} A_{k'}(\cosh kx)^{-k'}$ and $\Psi = \sum_{k'} B_{k'}(\cosh kx)^{-k'-1} \sinh kx$, and the corresponding eigenvalues of the energy are related to a band structure with a characteristic energy gap as a significant feature. In the present paper, it is shown that a further spectrum exists exhibiting the general structure $\Psi = \sum_{k'=0}^{\infty} A_{k'}(\cosh kx)^{-k'-1/2}$ and $\Psi = \sum_{k'=0}^{\infty} B_{k'}(\cosh kx)^{-k'-3/2} \sinh kx$ and yielding also a band structure. An extension of the solution spectrum to a nonlinear Klein-Gordon equation and a nonlinear Dirac equation does not imply essential difficulties, and the corresponding characteristic band structure has to be related to a mass spectrum.

1. INTRODUCTION

The solitary wave solutions

$$\Psi = A(\cosh kx)^{-1} \tag{1}$$

and

$$\Psi = B \tanh kx \tag{2}$$

of the stationary nonlinear Schrödinger equation

$$E\Psi + \frac{\hbar^2}{2m} \Delta\Psi = \lambda |\Psi|^2 \Psi \tag{3}$$

have been taken into consideration in many areas of physics, such as solid state and plasma physics (Auer, 1979; Kubo *et al.*, 1976; Satsuma and Yajima, 1974; Scott, 1973; Zakharov and Shabat, 1973), molecular and biophysics (Beaconsfield and Balanovski, 1984; Campbell and Peyrar, 1983; Carter, 1981; Davydov, 1976, 1979; Su *et al.*, 1980; Ulmer, 1988; Ulmer and

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Hartmann, 1978), and the time-dependent version of equation (3)

$$i\hbar\frac{\partial\Psi}{\partial t}+\frac{\hbar^2}{2m}\Delta\Psi=\lambda|\Psi|^2\Psi \tag{4}$$

and its relativistic extensions (e.g., a nonlinear type of Klein-Gordon equation) have also been studied with regard to the theory of measuring processes and elementary particle physics (Jackiw, 1977; Mielke, 1981; Mielnik, 1974; Barut, 1977).

In connection with the role of equation (3) in quasiparticle concepts in solid state and molecular physics, it has been pointed out (Ulmer, 1988) that this equation is completely equivalent to the Ginzburg-Landau theory of phase transitions (Landau and Ginzburg, 1950; Ginzburg, 1955, 1958), where the Lagrange density is given by

$$\mathscr{L}=(\alpha|\nabla\Psi|^2+\beta|\Psi|^2+\gamma|\Psi|^4) \tag{5}$$

This theory has been applied to superconductivity, and performing $\delta\mathscr{L}=0$ with respect to Ψ^* yields equation (3); the parameters α, β, γ have to be chosen appropriately. The relation between the Ginzburg-Landau theory (5) and the BCS theory has been discussed in detail (Gorkov, 1958), and it appears to be justified to regard the Cooper pairs as a specific kind of soliton. Because of the importance of equation (3) in a rather wide field of actual (and potential) applications, I have presented (Ulmer, 1988) an analysis of a solution spectrum of equation (3) representing generalizations of the solutions (1) and (2). Thus, a spectrum of symmetric (L_2-integrable) wave functions is given by

$$\begin{gathered}\Psi_\beta^{M\pm}=\sum_{k'=\beta}^{\infty}A_{k'}^{\beta,M\pm}(\cosh kx)^{-2k'+\beta}\\ E(\beta)=-\hbar^2k^2\beta^2/2m\\ \beta=1,2,3,\ldots\qquad\text{and}\qquad M=1,2,3,\ldots\end{gathered} \tag{6}$$

exhibiting the following additional properties: The degree of degeneracy is denumerably infinite ($M=1,2,3,\ldots$); the plus sign stands for soliton and the minus sign for antisoliton solution; and for every β ($\beta=1,2,3,\ldots$) the permitted k values are restricted within upper and lower boundaries

$$k^2_{\max}(\beta)>k^2>k^2_{\min}(\beta) \tag{7}$$

which actually can be related to a band structure.

The antisymmetric (L_2-integrable) solution spectrum is given by

$$\begin{gathered}\Psi_\beta^{M\pm}=\sum_{k'=\beta}^{\infty}B_{k'}^{\beta,M\pm}(\cosh kx)^{-(2k'-\beta+1)}\sinh kx\\ E(\beta)=-\frac{\hbar^2k^2}{2m}\beta^2\qquad(M=1,2,3,4,\ldots;\ \beta=1,2,3,\ldots)\end{gathered} \tag{8}$$

and the band structure condition (7) must also be valid with regard to the solution spectrum (8): $k_{\max}^2(\beta) > k^2 > k_{\min}^2(\beta)$ ($\beta = 1, 2, 3, \ldots$).

Thus, the band structure properties result from the convergence conditions of the Leibniz criterion of conditionally convergent series, which is closely related to the bound state condition $\lambda < 0$ implying sign $A_{k'} = -\text{sign } A_{k'+1}$ (Ulmer, 1988), and I shall return to this analysis in the subsequent investigation, where I present a further solution spectrum of equation (3) similar to (6) and (8) and also exhibiting band structure properties. However, before I continue this analysis, I should like to complete some aspects of the solutions (6) and (8). The solution manifold can be extended to three space coordinates, and the band structure condition (7) remains valid when the substitutions $x \to (x_1, x_2, x_3) = \mathbf{x}$ and $k \to (k_1, k_2, k_3) = \mathbf{k}$ have been carried out, inducing expressions of the form $(\cosh \mathbf{kx})^{-k'}$ in expansion (6) and $(\cosh \mathbf{kx})^{-k'} \sinh \mathbf{kx}$ in expansion (8), whereby the energy $E(\beta)$ is now given by

$$E(\beta) = -h^2 k^2 \beta^2 / 2m; \qquad k^2 = \mathbf{k}^2 = k_1^2 + k_2^2 + k_3^2; \quad \mathbf{k}_{\max}^2(\beta) > \mathbf{k}^2 > \mathbf{k}_{\min}^2(\beta) \tag{9}$$

However, the solution functions (6) and (8) are not the only L_2-integrable expansions obeying equation (3), and it will be shown that the modifications

$$\Psi = \sum_{k'=0}^{\infty} A_{k'} (\cosh kx)^{-k'-1/2} \tag{10}$$

and

$$\Psi = \sum_{k'=0}^{\infty} B_{k'} (\cosh kx)^{-k'-3/2} \sinh kx \tag{11}$$

also represent a solution spectrum of equation (3). Later it will be shown that other kids of expansions [e.g., on the basis of cosh $kx)^{-1/3}$] do not exist.

2. THE SOLUTION SPECTRUM OF THE EXPANSION $\Psi = \sum_{k'=0}^{\infty} A_{k'} (\cosh kx)^{-k'-1/2}$

With the help of the ansatz (1), equation (3) takes the form

$$\mathscr{E}_0 \sum_{k'=0}^{\infty} A_{k'} (\cosh kx)^{-(k'+1/2)} + \sigma k^2 \sum_{k'=0}^{\infty} A_{k'} (k' + \tfrac{1}{2})^2 (\cosh kx)^{-(k'+1/2)}$$

$$+ \sigma k^2 \sum_{k'=0}^{\infty} A_{k'} (k' + \tfrac{1}{2})(k' + \tfrac{3}{2}) (\cosh kx)^{-(k'+5/2)}$$

$$= \sum_{p,q,r=0}^{\infty} A_p A_p A_r (\cosh kx)^{-(3/2+p+q+r)} \tag{12}$$

whereby we make use of the substitutions $\mathscr{E}_0 = E/\lambda$, $\sigma = \hbar^2/2m\lambda$, and $u = \sigma k^2$.

Equation (12) must be satisfied for arbitrary values of the argument x, and this implies that the coefficients of each power of cosh kx occurring in equation (12) have to satisfy this equation. [This is also true for the previous solution functions (Ulmer, 1988), and therefore I do not repeat or reiterate the arguments.] Thus, equation (12) has to be analyzed with respect to each power of $(\cosh kx)^{-k'-1/2}$:

$k' = 0$ $[(\cosh kx)^{-1/2}]$ implies the relations

$$\mathscr{E}_0 A_0 + \tfrac{1}{4}\sigma k^2 A_0 = 0$$

(A_0 = arbitrary) and

$$\mathscr{E}_0 = -\sigma k^2/4 \tag{13}$$

or

$$E = -\hbar^2 k^2/8m$$

$k' = 1$ $[(\cosh kx)^{-3/2}]$ implies the relation

$$A_1 = A_0^3/2u \tag{13a}$$

because the energy has already been fixed by the relation (13). By considering the powers $k' = 2, 3, 4, 5, \ldots$ in equation (12), the expansion coefficients $A_2, A_3, \ldots$ of (10) can be determined recursively in terms of A_0: $A_0 \to A_1 \to A_2 \to A_3 \cdots \to A_n$. Table I gives the expansion coefficients A_n up to the order $n = 6$. Thus, the general formation law $A_n = A_n(A_0)$ is given by the recurrence formula

$$A_n = \sum_{p=0}^{M} \frac{A_0^{2n+1-4p}(2n+1-4p)\prod_{j=0}^{p}(2n+3-2j)}{2^n u_{n-2p} 2^p p!\,(2n+3)(2n+1)} \tag{14}$$

Table I. The A_0 Dependence of the Expansion Coefficients A_n $(n = 0, 1, \ldots, 6)^a$

n	A_n	A_0u^0	$A_0^3u^{-1}$	$A_0^5u^{-2}$	$A_0^7u^{-3}$	$A_0^9u^{-4}$	$A_0^{11}u^{-5}$	$A_0^{13}u^{-6}$
0	A_0	1	0	0	0	0	0	0
1	A_1	0	1/2	0	0	0	0	0
2	A_2	$1/2^3$	0	$1/2^2$	0	0	0	0
3	A_3	0	$3/2^4$	0	$1/2^3$	0	0	0
4	A_4	$7/2^7$	0	$5/2^5$	0	$1/2^4$	0	0
5	A_5	0	$27/2^8$	0	$7/2^6$	0	$1/2^5$	0
6	A_6	$33/2^{10}$	0	$55/2^9$	0	$9/2^7$	0	$1/2^6$

[a] $u = k^2\hbar^2/2m\lambda$. According to this table, A_6 is given by

$$A_6 = 33A_0/2^{10} + 55A_0^5/u^2 2^9 + 9A_0^9/u^4 2^7 + A_0^{13}/u^6 2^6.$$

where the following conditions must hold:

$$\begin{aligned} M &= n/2 && \text{if } n \text{ even} \\ M &= (n-1)/2 && \text{if } n \text{ odd} \end{aligned} \tag{14a}$$

This formula is required for an analysis of the convergence properties, and it can be constructed by an evaluation of equation (12), where the linear part of (3) is determined by the cubic term and, by that, the corresponding powers of the form $(\cosh kx)^{-(k'+1/2)}$ ($k' = 0, 1, 2, 3, \ldots$) have to be mutually compared with regard to coefficients of the nonlinear contributions:

$$\begin{aligned} &\text{linear contributions} \cdot (\cosh kx)^{-1/2-k'} \\ &\quad = \textstyle\sum_{p,q,r} A_p A_q A_r (\cosh kx)^{-1/2-(1+p+q+r)} \end{aligned}$$

where the condition $p+q+r+1 = k'$ must always hold. There is one combination with $p = q = r$, whereas there are three combinations $p = q \neq r$ (cyclic) and six combinations $p \neq q \neq r \neq p$ (cyclic). However, we have not yet defined A_0, because formula (14) only expressed the A_0 dependence of A_n with $n \geq 1$, and A_0 can be defined by the additional assumption of a norm. In order to be in agreement with the basis principles of (linear) quantum mechanics, I assume that the L_2-norm represents an adequate frame for the determination of A_0:

$$\|\Psi\|_2 = \int_{-\infty}^{+\infty} |\Psi|^2 \, dx = 1 \tag{15}$$

but I point out that the introduction of this norm implies the same difficulties as verified in the previous publication (Ulmer, 1988) with regard to the expansions (6) and (8), because the consideration of convergence problems now becomes intractable. The computation of the integrals $S_{m+n+1} = \int_{-\infty}^{+\infty} (\cosh kx)^{-1-n-m} \, dx$ is treated in the Appendix [they result from the norm condition (15)], and with the help of the substitution $\alpha = A_0^2$ we have to analyze the convergence properties of a polynomial equation of infinite degree:

$$c_1\alpha + c_2\alpha^2 + \cdots + c_n\alpha^n = 1, \qquad n \to \infty \tag{16}$$

which cannot be solved by algebraic means, and therefore we cannot use equation (16) for a proof of the convergence of the expansion (10) with respect to equation (3). However, equation (16) provides additional information concerning equation (3) and the expansion (10): If the existence of solutions of the form (10) is shown, then equation (16) induced by the L_2 norm yields a denumerably infinite set of A_0^2 values for each possible k value, and therefore the degree of degeneracy of the energy eigenvalue (13) is infinite, whereas A_0 itself may assume both signs (plus stands for soliton function and minus for antisoliton function).

Using the recurrence formula (14), I now show the existence of the pointwise convergence of the expansion (10) with respect to equation (12). For this purpose, it is also convenient to consider some special cases, e.g., the linear Schrödinger equation ($\lambda = 0$), which cannot be satisfied by the expansion (10). Thus, by taking $\lambda \to 0$ the equation (12) assumes the form

$$E \sum_{k'=0}^{\infty} A_{k'}(\cosh kx)^{-(k'+1/2)} = -\frac{\hbar^2 k^2}{2m} \sum_{k'=0}^{\infty} A_{k'}\left(\frac{k'+1}{2}\right)^2 (\cosh kx)^{-(k'+1/2)} + \frac{\hbar^2 k^2}{2m} \sum_{k'=0}^{\infty} A_{k'}\left(k'+\frac{1}{2}\right)\left(k'+\frac{3}{2}\right)^{-(k'+2+1/2)} \tag{17}$$

The energy E is equated to $E = -1 \cdot \hbar^2 \cdot k^2/(4 \cdot 2m)$, and by the substitution $\mathscr{C} = (\cosh kx)^{-1}$, equation (17) yields

$$\frac{A_{k'}}{A_{k'-2}} = \frac{(k'-3/2)(k'-1/2)}{k'(k'+1)} \cdot \mathscr{C}^2 \tag{18}$$

It is the zero point ($x = 0 \to \mathscr{C} = 1$) which causes the divergent behavior in the linear case, because for $\mathscr{C} = 1$ and $\lim k' \to \infty$ equation (18) converges to 1, but outside the zero point ($x \neq 0$, $\mathscr{C} < 1$) we obtain, according to equation (18), for $\lim k' \to \infty$, $A_{k'}/A_{k'-2} = \mathscr{C}^2 < 1$, and therefore the expansion (8) converges in the case of the linear Schrödinger equation ($\lambda = 0$) absolutely, except at the zero point $x = 0$. However, due to the nonlinear contributions with $\lambda \neq 0$ we are able to show that equation (12) exhibits pointwise convergence for $-\infty \leq x \leq +\infty$, but this property results from the relation $\lambda < 0$ (bound states) providing conditionally convergent series expansions.

It should be pointed out that it is sufficient to consider the convergence properties of equation (12) at the zero point ($x = 0$, $\mathscr{C} = 1$), because the expansion

$$\Psi = \sum_{k'=0}^{\infty} A_{k'} \mathscr{C}^{(k'+1/2)} \tag{19}$$

is convergent for $\mathscr{C} < 1$ if it shows convergence for $\mathscr{C} = 1$. The behavior of the expansion coefficients A_n ($n = 1, 2, 3, \ldots$) in terms of A_0 can be verified in Table I or formula (14): For each n ($n = 1, 2, \ldots$) A_n consists of odd powers of A_0 yielding throughout the same sign for each A_n as that of A_0, but with regard to u (recall that $u = \hbar^2 k^2/2m\lambda$) the same behavior is not true; so for all A_n with odd n only odd powers of u occur, whereas for all A_n with even n the powers of u are also even or, in other words, for $\lambda < 0$

(bound states) we obtain an expansion with alternating sign, and therefore the Leibniz criterion of conditionally convergent series is applicable:

$$\text{condition 1:} \qquad \sum_{n=0}^{\infty} A_n = \sum_{n=0}^{\infty} |A_n|(-1)^n \tag{20}$$

A second criterion for this kind of series is as follows:

$$\text{condition 2:} \qquad \lim_{n\to\infty} A_n \to 0 \tag{20a}$$

Thus, it can be verified from the relation (14) that the condition (20a) can be satisfied if the norm amplitude A_0 is finite, and even in the linear case the condition (20a) can be satisfied [equation (18)]. The third condition of the Leibniz criterion is somewhat more difficult, because it provides a band structure and stands in a close relationship to decision problems:

$$\text{condition 3:} \qquad |A_0| > |A_1| > |A_2| > \cdots > |A_n| \tag{20b}$$

Thus, the relation (20b) with regard to the formation law (14) implies a recursive function: As a first step, one has to determine the inequalities such that $|A_0| > |A_1| > |A_2| > |A_3|$ can be fulfilled for proper upper and lower bounds of u (or of k^2). As a second step, one has to use these bounds of k^2 and to check whether they are already sufficient for arbitrary A_n when we are passing to A_{n+1}. If they are not generally suitable, then the procedure has to be repeated by taking into account A_4, A_5, A_6, etc. With respect to equations (12) and (14), the relations

$$\begin{aligned} u_{\min} &= \tfrac{1}{2}A_0^2[4-(9-\delta)^{1/2}] \qquad (0<\delta) \\ u_{\max} &= \tfrac{1}{2}A_0^2[4+(9-\delta)^{1/2}] \qquad (\delta<g) \end{aligned} \tag{20c}$$

or

$$\underset{(0<\delta)}{2m|\lambda|\hbar^{-2}\frac{A_0^2}{2}[4-(9-\delta)^{1/2}]} < k^2 < \underset{(\delta<9)}{\frac{2m|\lambda|A_0^2}{2\hbar^2}[4+(9-\delta)^{1/2}]} \tag{20d}$$

turn out to be sufficient for the third Leibniz condition.

The further procedures are similar to those of Ulmer (1988): A_0 is only defined by the L_2 norm, but in order to establish the pointwise convergence, we have to assume that the norm parameter A_0 must also satisfy (20c) or (20d). However, the existence of the L_2 norm cannot be shown in a direct manner, e.g., via equation (16), but it follows from the existence of a maximum norm M_n and the L_1 norm $L_1(\Psi) = \int_{-\infty}^{+\infty} |\Psi|\, dx < \infty$, which themselves can be derived from the pointwise convergence (see the Appendix): Let $\Psi(x)$ be a continuous mapping on $\mathbb{R}$ with the properties

$$\lim_{|x|\to\infty} \Psi(x) = 0, \qquad \lim A_n \to 0 \qquad (n\to\infty) \tag{21}$$

and $\int_{-\infty}^{+\infty} |\Psi|\, dx < \infty$ and M_n also exist; then the inequality

$$|\Psi|^2/M_n^2 \leq |\Psi|/M_n \tag{22}$$

holds. From this inequality it follows that

$$\int_{-\infty}^{\infty} (|\Psi|/M_n)^2\, dx \leq \int_{-\infty}^{\infty} (|\Psi|/M_n)\, dx \tag{22a}$$

is also true, yielding $\int_{-\infty}^{+\infty} |\Psi|^2\, dx \leq M_n \int_{-\infty}^{+\infty} |\Psi|\, dx$. With regard to equation (16), where we are only able to solve polynomial equations of finite order by numerical methods, e.g., $M = 100$, see Ulmer (1988) concerning the expansions (8) and (10), but these remarks do not touch the principal existence of solutions of (3).

However, the present analysis has only considered a very interesting standard case, where A_0 represents the norm amplitude, but equation (12) can also be discussed under some modified conditions: Assume $A_0 \equiv 0$; A_1 now assumes the role of the norm amplitude. Then the energy E is equated to $E_1 = -(\hbar^2 k^2/2m)(1/4+2)$ with the additional condition that according to equation (12) the subsequent expansion coefficients $A_2, A_3, \ldots, A_n$ are defined in terms of A_1: $A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_n$. The discussion of convergence properties is the same as already presented. It is also possible to start in equation (12) with an arbitrary A_β, then those expansion coefficients $A_{k'}$ ($k' < \beta$) have to be put $\equiv 0$, and equation (12) has to be discussed with regard to the expansion

$$\Psi_\beta = \sum_{k'=\beta}^{\infty} A_{k'} (\cosh kx)^{-(2k'-\beta+1/2)} \tag{23}$$

This general case is also closely related to the standard case, where the expansion coefficients run from $\beta = 0$, and the norm is now defined in terms of A_β. However, equation (16) holds in an analogous manner, and therefore each eigenfunction Ψ_β exhibits an infinite degree of degeneracy:

$$\Psi_\beta^{M\pm} = \sum_{k'=\beta}^{\infty} A_{k'}^{M\pm} (\cosh kx)^{-(2k'-\beta+1/2)} \qquad M = 1, 2, 3, \ldots; \quad \beta = 0, 1, 2, 3, \ldots \tag{24}$$

The energy spectrum E_β according to equation (12) is given by

$$E(\beta) = -\hbar^2 k^2 [\beta(\beta+1) + 1/4]/2m \qquad k_{\max}^2 > k^2 > k_{\min}^2 \tag{24a}$$

Thus, the convergence properties are not different from those discussed in this analysis, and with the help of proper substitutions and estimations all convergence aspects can be reduced to those of the standard case. The same fact is also true for the antisymmetric eigenfunctions presented in the next section. Some further remarks complete this section:

1. The transition to three dimensions $x \to \mathbf{x}$ is also straightforward. We have to replace kx by $\mathbf{k} \cdot \mathbf{x}$ in all relevant equations [e.g., equation (12)], and relation (24a) now reads

$$E(\beta) = -\hbar^2 k^2[\beta(\beta+1)+1/4]|2m$$
$$k^2 = k_1^2 + k_2^2 + k_3^2; \qquad k_{\max}^2 > k^2 > k_{\min}^2 \tag{24b}$$

2. The solution functions (24) obeying equation (3) can be made time dependent by Galilei-transformed wave functions

$$\Psi' = \exp[imvx/\hbar - i(E + mv^2/2)t/\hbar]\Psi(x - vt) \tag{25}$$

where Ψ' obeys solely equation (4).

3. There arises also the question of whether other kinds of powers, e.g., $(\cosh kx)^{-1/4}$, $(\cosh kx)^{-5/4}$, etc., can also satisfy equation (3). But this is really not true and may be readily verified by the general ansatz

$$\Psi = \sum_{k'=0}^{\infty} A_{k'}(\cosh kx)^{-(k'+\rho)} \tag{26}$$

which has to be substituted into equation (3) or (12). Because equation (3) has to be satisfied with respect to each power of $(\cosh kx)^{-(k'+\rho)}$ on both sides (linear and nonlinear terms) and for arbitrary arguments $-\infty \leq x \leq +\infty$, it follows that ρ may assume either integer numbers or half-odd numbers:

$$\begin{aligned}
&(\text{linear terms}) \cdot (\cosh kx)^{-k'-\rho} \\
&\quad = (\cosh kx)^{-3\rho-p-q-r} \cdot (\text{nonlinear terms}) \\
&\quad \Rightarrow k' + \rho = 3\rho + m \qquad (m = p + q + r) \\
&\qquad k' - m = 2\rho \to \rho = 1/2, 1, 3/2, 2, \ldots \\
&\qquad k' - m = 1, 2, 3, 4, \ldots
\end{aligned}$$

The first case is related to the already discussed expansions (6) and (8), whereas the second case refers to the expansions (10) and (11). Using the general ansatz (26), we should be able to show that ρ depends on the degree of nonlinearity. If the lowest nonlinear term (besides the linear contributions) is of the order 5, e.g., a nonlinear Schrödinger equation of the form

$$E\Psi + \frac{\hbar^2}{2m}\Delta\Psi = \lambda|\Psi|^4\Psi \tag{27}$$

then ρ could assume integer and half-odd values, but also values of the form odd/4. On the other hand, a nonlinearity of the form $\lambda_1|\Psi|^2\Psi + \lambda_2|\Psi|^4\Psi$ would only permit $\rho =$ integer and half-odd, but not odd/4.

3. THE SOLUTION SPECTRUM OF THE EXPANSION $\Psi = \sum_{k'=0}^{\infty} B_{k'}(\cosh kx)^{-k'-3/2} \sinh kx$

It has already been indicated by equation (11) that the expansion (10) is not the only possibility to satisfy equation (3) by L_2-integrable wave functions. Thus, the antisymmetric set of wave functions given by equation (11) can be regarded as a modification of the expansion (8) or as a generalization of the solution (2), but the latter solution function is not square-integrable. Substituting the expansion (10) into equation (3), we obtain

$$
\begin{aligned}
& E \sum_{k'=0}^{\infty} B_{k'}(\cosh kx)^{-k'-2/3} \sin hkx \\
& + \frac{\hbar^2 k^2}{2m} \sum_{k'=0}^{\infty} B_{k'}\left(-k'+\frac{1}{2}\right)^2 (\cosh kx)^{-k'-3/2} \sinh kx \\
& - \frac{\hbar^2 k^2}{2m} \sum_{k'=0}^{\infty} B_{k'}\left(-k'-\frac{1}{2}\right)\left(-k'-\frac{3}{2}\right)(\cosh kx)^{-k'-7/2} \sinh kx \\
& = \lambda \sum_{p,q,r=0}^{\infty} B_p B_q B_r \sinh kx \cdot (\cosh kx)^{-(9/2+p+q+r)}(\cosh^2 kx - 1) \qquad (28)
\end{aligned}
$$

It can be verified from equation (28) that both on the left-hand side (linear terms) and on the right-hand side (nonlinear contributions) we have to regard polynomials of the form $(\cosh kx)^{-m-3/2} \sinh kx$. Therefore, the analysis of equation (28) completely corresponds to that of equation (12), and the norm amplitude of the first eigenfunction now is B_0. Thus, the whole procedure of the preceding section is applicable: All the subsequent expansion coefficients B_n $(n \geq 1)$ are defined in terms of B_0, which is itself only fixed by the L_2-norm [e.g., equation (15)], and equation (16) also holds in an analogous fashion. The B_0 dependence of B_1 is easy to see from equation (28), and is $B_1 = B_0^3/2u$ $(u = \hbar^2 k^2/2m\lambda)$. But it is also possible to put $B_{k'} \equiv 0$ for $k' < \beta$ $(\beta = 0, 1, 2, 3, \ldots)$; then, from equation (28) follows the existence of a spectrum of eigenfunctions and eigenvalues, where each eigenfunction $(\beta = 0, 1, 2, 3, \ldots)$ additionally exhibits an infinite degree of degeneracy $(M = 1, 2, 3, \ldots)$:

$$
\Psi_\beta^{M\pm} = \sum_{k'=\beta}^{\infty} B_{k'}^{M\pm}(\cosh kx)^{-(2k'-\beta+3/2)} \sinh kx \qquad (29)
$$

$$
\beta = 0, 1, 2, \ldots; \quad M = 1, 2, \ldots
$$

The energy spectrum related to the expansion (27) is given by

$$
E(\beta) = -\frac{\hbar^2 k^2}{2m}[\beta(\beta+1) + 1/4] \qquad (30)
$$

However, the boundaries of the permitted k^2 values are also given an inequality:

$$k^2_{\min}(\beta) < k^2 < k^2_{\max}(\beta) \tag{30a}$$

This inequality is a consequence of the fact that for $\lambda < 0$, by the determination of $B_{k'}$ in terms of B_β $(k' > \beta)$, a conditionally convergent series results and the conclusions according to the preceding section concerning the Leibniz criterion also hold. Equation (28) does not satisfy the linear Schrödinger equation ($\lambda = 0$), although it may appear that the antisymmetric expansion (11) does not provide significant difficulties at the zero point $(x = 0)$, because the expansion (11) or (29) vanishes at $x = 0$, whereas $(\cosh kx)^{-k'-1/2}$ is 1 at $x = 0$ (for all $k' \geq 0$). Defining

$$f_\beta = (\cosh kx)^{-\beta-3/2} \sinh kx \qquad (\beta = 0, 1, 2, \ldots) \tag{31}$$

we can verify that $f_\beta(x)$ exhibits a maximum at

$$x_{\max} = |k|^{-1} ar \cosh[+(\beta + 1/2)/(\beta + 3/2)] \tag{31a}$$

and a minimum at

$$x_{\min} = |k|^{-1} ar \cosh[-(\beta + 1/2)/(\beta + 3/2)] \tag{31b}$$

but by taking $\lim \beta \to \infty$, f_β is zero everywhere and produces a jump from -1 to $+1$ at the zero point, and this discontinuity is the reason that equation (28) can only be solved in the nonlinear case with $\lambda < 0$, yielding conditionally convergent expansions and a band structure for the corresponding k^2 values. However, there may be interesting aspects with regard to the antisymmetric L_2-integrable wave functions given by the expansions (8) and (11), because these soliton functions represent dipole soliton functions: Thus, recently (Beaconsfield and Balanovski, 1984) dipole solitons have been considered in molecular biology (e.g., DNA replication), and it appears that the transport of dipole solitons in long molecular chains is an exciting concept in many problems of molecular electronics (Campbell and Peyrar, 1983; Carter, 1981). A characteristic feature of the solution functions (10) and (11) is that they are related to an energy gap, as known by superconductivity and other kinds of phase transitions [the equivalence between the Ginzburg–Landau theory (5) and equation (3) has already been pointed out].

The transition to the linear Schrödinger equation is, as already mentioned, not possible by equations (12) and (28), and even by a linear combination of the expansions (10) and (11) we cannot pass to the linear case. However, by the ansatz

$$\Psi_\beta = \exp(i\rho x) \cdot \left[\sum_{k'=\beta}^{\infty} A_{k'}^{M\pm} (\cosh kx)^{-(2k'-\beta+1/2)} + B_{k'}^{M\pm} (\cosh kx)^{-(2k'-\beta+3/2)} \sinh kx \right] \tag{32}$$

the corresponding transition can be carried out, where the energy spectrum now is given by

$$E(\beta) = -\frac{(\hbar^2 k^2 - \hbar^2 \rho^2)[\beta(\beta+1)+1/4]}{2m}$$

$$k_{\min}^2 < (k^2 - \rho^2) < k_{\max}^2 \tag{30a}$$

On the other hand, the ansatz (32) implies that we cannot distinguish between symmetric and antisymmetric functions, and therefore such a transition to the case $\lambda = 0$ must be associated with a breaking of the symmetry.

4. RELATIVISTIC EXTENSIONS

With the help of slight modifications, the eigenfunctions (10) and (11) become solutions of a nonlinear Klein-Gordon equation

$$\Box\Psi = \frac{m^2 c^2}{\hbar^2}\Psi + \lambda|\Psi|^2\Psi \tag{33}$$

In the same fashion as in the nonrelativistic case the problems of convergence also have to be solved with regard to equation (33), and here I only state briefly the most essential results: The relativistic analog of (10) is ($M = 1, 2, 3, \ldots$; $\beta = 0, 1, 2, 3, \ldots$)

$$\Psi_\beta^{M\pm} = \sum_{k'=\beta}^{\infty} A_{k'}^{M\pm}[\cosh(\gamma kx - \gamma kvt]^{-(2k'-\beta+1/2)} \tag{34}$$

where the mass spectrum is given by $[\gamma = 1/(1 - v^2/c^2)^{1/2}]$

$$m^2 = \frac{\hbar^2 k^2}{c^2}[\beta(\beta+1)+1/4], \qquad \beta = 0, 1, 2, 3, \ldots \tag{35}$$

and the relativistic analog of (11) now becomes

$$\Psi_\beta^{M\pm} = \sum_{k'=\beta}^{\infty} B_{k'}^{M\pm}[\cosh(\gamma kx - \gamma kvt]^{-(2k'-\beta+3/2)}$$

$$\times \sinh(\gamma kx - \gamma kvt) \tag{36}$$

from which the mass spectrum is identical to that in (35):

$$m^2 = \frac{\hbar^2 k^2}{c^2}[\beta(\beta+1)+1/4], \qquad \beta = 0, 1, 2, 3, \ldots \tag{37}$$

Due to the conditionally convergent behavior of the expansions (34) and (36) we also obtain a band structure with regard to the mass spectrum:

$$k_{\min}^2(\beta) < k^2 < k_{\max}^2(\beta) \tag{38}$$

It is evident that the above-mentioned expansions cannot represent eigenfunctions of the linear Klein-Gordon equation, but the corresponding transition to the case $\lambda = 0$ can be carried out if we admit that the symmetry may become broken, as expressed by the ansatz

$$\Psi_{\beta}^{M\pm} = \exp(ipx - i\omega t)$$
$$\times \sum_{k'=\beta}^{\infty} \{A_{k'}^{M\pm}[\cosh(k\gamma x - k\gamma vt)]^{-(2k'-\beta+1/2)}$$
$$+ B_{k'}^{M\pm}[\cosh(k\gamma x - k\gamma vt)]^{-(2k'-\beta+3/2)} \sinh(k\gamma x - k\gamma vt)\} \qquad (39)$$

The conditions according to the relations (35), (37) and (38) now read

$$m^2 = \frac{\hbar^2(k^2 - 4p^2 + 4\omega^2/c^2)}{c^2}\left[\beta(\beta+1) + \frac{1}{4}\right]$$
$$k_{\min}^2(\beta) < (k^2 - 4p^2 + 4\omega^2/c^2) < k_{\max}^2(\beta) \qquad (40)$$

With respect to equation (33), see Jackiw (1977) and Mielke (1981) for further information and references.

It is also possible to solve nonlinear spinor equations using the expansions (34) and (36), but due to the spin-spin coupling inducing a difficult multiplet structure, the effort for the evaluation of nonlinear terms increases. Yet this increasing effort does not imply any significant difficulty, and all conclusions concerning the band structure properties and conditionally convergent expansions also hold in an appropriate form. Nonlinear spinor equations have been proposed by some authors (Heisenberg, 1966; Ivanenko, 1979; and references cited therein). As in the previous paper, I only consider a simplified model case, e.g., a restriction of the four-component Dirac spinor Ψ to $(\Psi_1, 0, 0, 0)$:

$$\gamma^{\nu}\frac{\partial\Psi}{\partial x^{\nu}} = \frac{mc}{\hbar}\Psi + \lambda\Psi(\bar{\Psi}\gamma_5\Psi) \qquad (41)$$

which takes the form (special case: coordinates z and t)

$$-\gamma_z\hbar\frac{\partial\Psi_1}{\partial z} + \frac{\hbar}{c}\gamma_t\frac{\partial\Psi_1}{\partial t} = mc\Psi_1 + \lambda\hbar\Psi_1^3 \qquad (42)$$

Then, by the expansions

$$\Psi_{1,\beta}^{M\pm} = \sum_{k'=\beta}^{\infty} [A_{k'}^{M\pm}[\cosh(k_1\gamma_z z - k_4\gamma_t t)]^{-(2k'-\beta+1/2)}$$
$$+ B_{k'}^{M\pm}[\cosh(k_1\gamma_z z - k_4\gamma_t t)]^{-(2k'-\beta+3/2)} \sinh(k_1\gamma_z z - k_4\gamma_t t) \qquad (43)$$

we can solve equation (42) with the help of the principles as above, yielding

$$m = (\hbar/c)(\beta + 1/2)(k_1\gamma_z + k_4\gamma_t)$$
$$m^2 = (\hbar^2/c^2)[\beta(\beta+1)+1/4][(k_1^2 - k_4^2)] \tag{44}$$

and

$$k_{\min}^2(\beta) < (k_1^2 - k_4^2) < k_{\max}^2(\beta) \tag{44a}$$

The transition to the linear Dirac equation can only be performed by the modification according to the ansatz (39). These considerations may be a clear indication that perturbation methods, starting with the linear case, are often insufficient or they may fail.

APPENDIX: SOME COMPUTATIONAL ASPECTS

With respect to the evaluation of the equation (16), which results from the introduction of the L_2 norm, the following expressions have to be regarded: $\|\Psi\|_2 = 1$ implies

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_n A_m S_{1+m+n} = 1 \tag{A1}$$

where $S_{1+m+n} = \int_{-\infty}^{+\infty} dx\,(\cosh kx)^{-(1+m+n)}$. If $1+m+n$ is odd, then it is given by

$$S_p = S_1 \frac{1\cdot 3\cdot 5\cdots(p-2)}{2\cdot 4\cdot 6\cdots(p-1)}; \qquad S_1 = \frac{\pi}{k} \qquad (p = 1+m+n) \tag{A2}$$

and if $1+m+n$ is even, then it is determined by

$$S_p = S_2 \frac{2\cdot 4\cdot 6\cdots(p-2)}{3\cdot 5\cdot 7\cdots(p-1)}; \qquad S_2 = \frac{2}{k} \qquad (p = 1+m+n) \tag{A3}$$

In the preceding analysis, we have made use of the L_1 norm to show the existence of square-integrable solutions: $\int_{-\infty}^{+\infty} |\Psi(x)|\,dx < \infty$ implies the evaluation of the following integrals:

$$S_{1/2+m} = \int_{-\infty}^{+\infty} (\cosh kx)^{-1/2-m}\,dx \qquad (m = 0, 1, 2, \ldots) \tag{A4}$$

Thus, for even m, the integral (A4) can be brought to the form

$$S_{1/2+m} = S_{1/2} \frac{1\cdot 5\cdot 9\cdot (2m-3)}{3\cdot 7\cdot 11\cdot (2m-1)} \tag{A5}$$

where $S_{1/2}=5.0696/k$, and for odd m we obtain

$$S_{3/2+m}=S_{3/2}\frac{3\cdot 7\cdot 11\cdot(2m-1)}{5\cdot 9\cdot 13\cdot(2m+1)} \tag{A6}$$

where $S_{3/2}=2.3964/k$. Both $S_{1/2}$ and $S_{3/2}$ can be evaluated by the substitution $\cosh kx=1/\sin^2\varphi$, yielding integrals of the form

$$S_{1/2}=\frac{4}{k}\int_0^{\pi/2}(1+\sin^2\varphi)^{-1/2}\,d\varphi$$

$$S_{3/2}=\frac{4}{k}\int_0^{\pi/2}\sin^2\varphi\,(1+\sin^2\varphi)^{-1/2}\,d\varphi$$

With respect to convergence estimations, note that the following inequality holds:

$$S_{1/2}>S_1>S_{3/2}>S_2>S_{5/2}>\cdots>S_{(m+1)/2}\qquad (m>4)$$

and for $m\to\infty$, we obtain $\lim S_{(m+1)/2}\to 0$.

As already shown in Section 2, the existence of the L_2 norm of the expansion (8) with regard to equation (3) and (and also to other equations, e.g., the nonlinear Klein-Gordon equation) can be reduced to the existence of a maximum norm M_n and an L_1 norm.

Maximum Norm M_n

Assume $M_n=\max|\Psi|$; then, for all expansions constructed on the basis (10) [a similar fact is also true for the expansion (11)] we have $M_n=\max|\Psi|=|\sum_{k'}A_{k'}|$, because the set $\{(\cosh kx)^{-k'}\}$ always exhibits a maximum at the zero point $x=0$ $(k'\geq 1/2)$. In the nonlinear case, the sum $\sum_{k'}A_{k'}$ holds, and therefore $M_n=|\sum_{k'}A_{k'}|<\infty$ is also true, where $\sum_{k'}|A_{k'}|<\infty$ does not exist.

With regard to the antisymmetric expansion (11), the maximum is usually not at the zero point, but at $x_m=k^{-1}ar\cosh[(k'+1/2)/(k'+3/2)]$, and therefore the existence of a maximum norm requires the evaluation of a modified sum:

$$M_n=\left|\sum_{k'=0}^{\infty}B_{k'}[(k'+1/2)/(k'+3/2)]^{-(k'/2+3/4)}\right|$$

With $\alpha=(2k'+1)/2$, we obtain

$$M_n=\left|\sum_{k'}B_{k'}\left(1+\frac{1}{\alpha}\right)^{\alpha/2}\left(1+\frac{1}{\alpha}\right)^{1/2}\right|\Rightarrow M_n\leq\left|\sum_{k'}B_{k'}\right|\cdot(3e)^{1/2}$$

L_1-Norm

The existence of

$$L_1(\Psi) = \int_{-\infty}^{+\infty} |\Psi(x)| \, dx < \infty$$

$$\Rightarrow L_1(\Psi) = \left| A_0 S_{1/2} + A_1 S_{3/2} + \cdots + \sum_m A_m S_{m+1/2} \right| < \infty \tag{A7}$$

follows from $|\sum_{k'} A_{k'}| < \infty$, because we can write

$$L_1(\Psi) = S_{1/2} |A_0 + (A_1 S_{3/2} + \cdots) S_{1/2}^{-1}{}' \leq S_{1/2} M_n$$

where $S_{n_1} > S_{n_2}$ (for $n_1 < n_2$) with $S_{n'} \to 0$ ($n' \to \infty$). Therefore we obtain

$$\int_{-\infty}^{\infty} |\Psi| \, dx \leq S_{1/2} M_n \tag{A8}$$

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